Chapter review 1

1 a $\sinh(\ln 3) = \dfrac{e^{\ln 3} - e^{-\ln 3}}{2}$

$= \dfrac{3 - \frac{1}{3}}{2} = \dfrac{4}{3}$

$e^{\ln 3} = 3$, and $e^{-\ln 3} = e^{\ln 3^{-1}} = \frac{1}{3}$

b $\cosh(\ln 5) = \dfrac{e^{\ln 5} + e^{-\ln 5}}{2}$

$= \dfrac{5 + \frac{1}{5}}{2} = \dfrac{13}{5}$

$e^{\ln 5} = 5$, and $e^{-\ln 5} = e^{\ln 5^{-1}} = \frac{1}{5}$

c $\tanh\left(\ln \dfrac{1}{4}\right) = \dfrac{e^{2\ln\frac{1}{4}} - 1}{e^{2\ln\frac{1}{4}} + 1}$

$= \dfrac{\left(\frac{1}{16} - 1\right)}{\left(\frac{1}{16} + 1\right)}$

$= -\dfrac{15}{17}$

$e^{2\ln\frac{1}{4}} = e^{\ln\left(\frac{1}{4}\right)^2} = \frac{1}{16}$

2 $\operatorname{artanh} x - \operatorname{artanh} y$

$= \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) - \dfrac{1}{2}\ln\left(\dfrac{1+y}{1-y}\right)$

Use $\ln a - \ln b = \ln\left(\dfrac{a}{b}\right)$

$= \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x} \times \dfrac{1-y}{1+y}\right)$

$= \dfrac{1}{2}\ln\left(\dfrac{1+x-y-xy}{1-x+y-xy}\right)$

Use $\frac{1}{2}\ln a = \ln a^{\frac{1}{2}}$

$= \ln\sqrt{\left(\dfrac{1+x-y-xy}{1-x+y-xy}\right)}$

So $\sqrt{\left(\dfrac{1+x-y-xy}{1-x+y-xy}\right)} = 5$

$\dfrac{1+x-y-xy}{1-x+y-xy} = 25$

$1 + x - y - xy = 25 - 25x + 25y - 25xy$

$24xy - 26y = 24 - 26x$

$y(12x - 13) = 12 - 13x$

$y = \dfrac{12 - 13x}{12x - 13}$

3 RHS $= \sinh A \cosh B - \cosh A \sinh B$

$$= \left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^B + e^{-B}}{2}\right) - \left(\frac{e^A + e^{-A}}{2}\right)\left(\frac{e^B - e^{-B}}{2}\right)$$

$$= \frac{e^{A+B} - e^{-A+B} + e^{A-B} - e^{-A-B}}{4} - \frac{e^{A+B} + e^{-A+B} - e^{A-B} - e^{-A-B}}{4}$$

$$= \frac{2(e^{A-B} - e^{-A+B})}{4}$$

$$= \frac{e^{A-B} - e^{-(A-B)}}{2}$$

$$= \sinh(A - B) = \text{LHS}$$

4 RHS $= \dfrac{2\tanh\frac{1}{2}x}{1 - \tanh^2\frac{1}{2}x}$

$$2\tanh\frac{1}{2}x = \frac{2(e^x - 1)}{e^x + 1}$$

$$1 - \tanh^2\frac{1}{2}x = 1 - \left(\frac{e^x - 1}{e^x + 1}\right)^2$$

$$= \frac{(e^x + 1)^2 - (e^x - 1)^2}{(e^x + 1)^2}$$

$$= \frac{4e^x}{(e^x + 1)^2}$$

So RHS $= \dfrac{2(e^x - 1)}{e^x + 1} \times \dfrac{(e^x + 1)^2}{4e^x}$

$$= \frac{(e^x - 1)(e^x + 1)}{2e^x}$$

$$= \frac{e^{2x} - 1}{2e^x}$$

$$= \frac{e^x - e^{-x}}{2}$$

$$= \sinh x = \text{LHS}$$

5 $9\cosh x - 5\sinh x = 15$

$$9\frac{\left(e^{x}+e^{-x}\right)}{2}-5\frac{\left(e^{x}-e^{-x}\right)}{2}=15$$

$$9e^{x}+9e^{-x}-5e^{x}+5e^{-x}=30$$

$$4e^{x}-30+14e^{-x}=0$$

$$2e^{x}-15+7e^{-x}=0$$

Multiply throughout by e^x.

$$2e^{2x}-15e^{x}+7=0$$

$$(2e^{x}-1)(e^{x}-7)=0$$

Solve as a quadratic in e^x.

$$e^{x}=\tfrac{1}{2},\ e^{x}=7$$

$$x=\ln\left(\tfrac{1}{2}\right),\ x=\ln 7$$

6 $23\sinh x - 17\cosh x + 7 = 0$

$$23\frac{(e^{x}-e^{-x})}{2}-17\frac{(e^{x}+e^{-x})}{2}+7=0$$

$$23e^{x}-23e^{-x}-17e^{x}-17e^{-x}+14=0$$

$$6e^{x}+14-40e^{-x}=0$$

$$3e^{x}+7-20e^{-x}=0$$

$$3e^{2x}+7e^{x}-20=0$$

Multiply throughout by e^x.

$$(3e^{x}-5)(e^{x}+4)=0$$

$$e^{x}=\tfrac{5}{3}$$

$e^{x}=-4$ is not possible for real x.

$$x=\ln\left(\tfrac{5}{3}\right)$$

7 $3\cosh^{2}x+11\sinh x=17$

Using $\cosh^{2}x-\sinh^{2}x=1$

$$3(1+\sinh^{2}x)+11\sinh x=17$$

$$3\sinh^{2}x+11\sinh x-14=0$$

$$(3\sinh x+14)(\sinh x-1)=0$$

$$\sinh x=-\frac{14}{3},\ \sinh x=1$$

$$x=\operatorname{arsinh}\left(-\frac{14}{3}\right),x=\operatorname{arsinh}1$$

Use $\operatorname{arsinh}x=\ln(x+\sqrt{x^{2}+1})$.

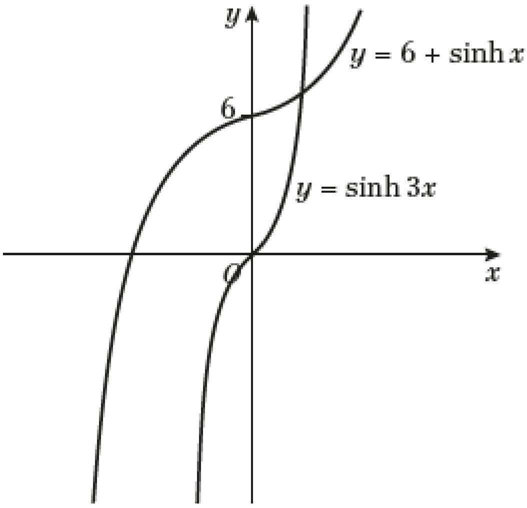
$$x=\ln\left(-\frac{14}{3}+\sqrt{\frac{196}{9}+1}\right)$$

$$=\ln\left(\frac{-14+\sqrt{205}}{3}\right)$$

$$x=\ln(1+\sqrt{1+1})$$

$$=\ln(1+\sqrt{2})$$

8 a



b At the intersection,

$$6+\sinh x=\sinh 3x$$
$$6+\sinh x=3\sinh x+4\sinh^3 x$$
$$4\sinh^3 x+2\sinh x-6=0$$
$$2\sinh^3 x+\sinh x-3=0$$
$$(\sinh x-1)(2\sinh^2 x+2\sinh x+3)=0$$

You can see, by inspection that $\sinh x=1$ satisfies this equation.

The equation $2\sinh^2 x+2\sinh x+3=0$ has no real roots, because $b^2-4ac=4-24<0$.

The only intersection is where $\sinh x=1$

For $\sinh x=1$,

$$x=\text{arsinh}1$$
$$=\ln\left(1+\sqrt{1^2+1}\right)$$
$$=\ln\left(1+\sqrt{2}\right)$$

Using $y=6+\sinh x$

$y=7$

Coordinates of the point of intersection are $\left(\ln\left(1+\sqrt{2}\right),\ 7\right)$

9 a $13\cosh x + 5\sinh x = R\cosh x\cosh\alpha + R\sinh x\sinh\alpha$

So $R\cosh\alpha = 13$

$R\sinh\alpha = 5$

$R^2\cosh^2\alpha - R^2\sinh^2\alpha = 13^2 - 5^2$ ← Use the identity $\cosh^2 A - \sinh^2 A = 1$.

$R^2\left(\cosh^2\alpha - \sinh^2\alpha\right) = 144$

$R^2 = 144$

$R = 12$

$\frac{R\sinh\alpha}{R\cosh\alpha} = \frac{5}{13}$

$\tanh\alpha = \frac{5}{13}$

$\alpha = 0.405$ ← Direct from calculator.

b $13\cosh x + 5\sinh x = 12\cosh(x + 0.405)$ ← For any value A, $\cosh A \geqslant 1$.

The minimum value of $13\cosh x + 5\sinh x$ is 12.

10 a $3\cosh x + 5\sinh x = R\sinh x\cosh\alpha + R\cosh x\sinh\alpha$

So $R\cosh\alpha = 5$

$R\sinh\alpha = 3$

$R^2\cosh^2\alpha - R^2\sinh^2\alpha = 5^2 - 3^2$ ← Use the identity $\cosh^2 A - \sinh^2 A = 1$.

$R^2\left(\cosh^2\alpha - \sinh^2\alpha\right) = 16$

$R^2 = 16$

$R = 4$

$\frac{R\sinh\alpha}{R\cosh\alpha} = \frac{3}{5}$

$\tanh\alpha = \frac{3}{5}$

$\alpha = 0.693$ ← Direct from calculator.

$3\cosh x + 5\sinh x = 4\sinh(x + 0.693)$

b $4\sinh(x + 0.693) = 8$

$\sinh(x + 0.693) = 2$

$x + 0.693 = \text{arsinh}2$

$= 1.44 \quad (3 \text{ s.f.})$ ← Direct from calculator.

$x = 0.75 \quad (2 \text{ d.p.})$

10 c $3\cosh x + 5\sinh x = 8$

$$3\frac{(e^x + e^{-x})}{2} + 5\frac{(e^x - e^{-x})}{2} = 8$$

$$3e^x + 3e^{-x} + 5e^x - 5e^{-x} = 16$$

$$8e^x - 16 - 2e^{-x} = 0$$

$$4e^x - 8 - e^{-x} = 0$$

Multiply throughout by e^x.

$$4e^{2x} - 8e^x - 1 = 0$$

Solve as a quadratic in e^x.

$$e^x = \frac{8 \pm \sqrt{64 + 16}}{8}$$

$$e^x = 1 \pm \frac{\sqrt{80}}{8} = 1 \pm \frac{\sqrt{5}}{2}$$

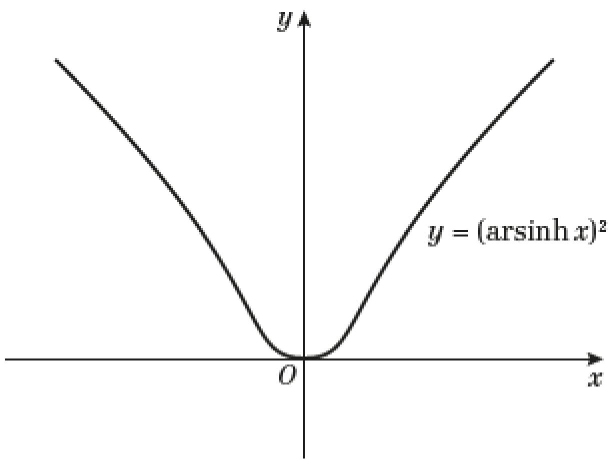
$$e^x = 1 + \frac{\sqrt{5}}{2}$$

$e^x = 1 - \frac{\sqrt{5}}{2}$ is negative, so not possible for real x.

$$x = \ln\left(1 + \frac{\sqrt{5}}{2}\right)$$

$$= 0.75 \quad (2 \text{ d.p.})$$

Challenge



$y = (\operatorname{arsinh} x)^2$